Solving a 3x3 System of Equations using Cramer's Rule

Consider the system of equations:

$$\begin{aligned} 3x + 2y + \ \ z &= \ \ 4 \\ 2x - 3y + 3z &= \ \ 6 \\ x + 4y - \ \ z &= -5 \end{aligned}$$

In matrix form, the system can be written:

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & -3 & 3 \\ 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -5 \end{bmatrix}$$

In short: $\underline{\mathbf{A}}\mathbf{x} = \mathbf{b}$, where $\underline{\mathbf{A}}$ is the coefficient matrix, $\mathbf{x}$ is the column vector of variables, and $\mathbf{b}$ is the column vector of constants.

Cramer's Rule 3x3

Step. 1 Calculate the Determinant Δ of the Coefficient Matrix

This method of taking the determinant works <u>only</u> for a 3x3 matrix system (not for a 4x4 and above).

$$\Delta = \det \begin{bmatrix} 3 & 2 & 1 \\ 2 & -3 & 3 \\ 1 & 4 & -1 \end{bmatrix} = \begin{vmatrix} 3 & 2 & 1 \\ 2 & -3 & 3 \\ 1 & 4 & -1 \end{vmatrix}$$

$$\begin{aligned} &= +(3)(-3)(-1) + (2)(3)(1) + (1)(2)(4) - (1)(-3)(1) - (4)(3)(3) - (-1)(2)(2) \\ &= +(9) + \ (6) + (8) - (-3) - (36) - (-4) \\ &= \ -6 \end{aligned}$$

Note that the matrix is written in square brackets [ ]; its determinant is written within "absolute value" lines | | .

The determinant for a 3x3 (and only for a 3x3) can be calculated as follows.

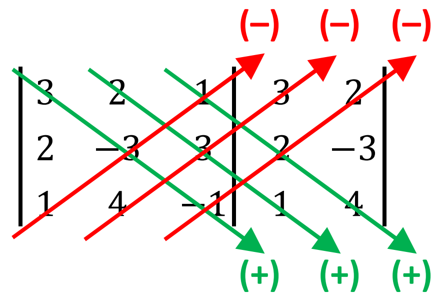
a. Write the 3x3 in its determinant form (within | |).
b. Write the first two columns to the right of the 3x3; this makes the process easier (but is not necessarily needed once you have practice).

$$\left| \begin{matrix} 3 & 2 & 1 \\ 2 & -3 & 3 \\ 1 & 4 & -1 \end{matrix} \right| \begin{matrix} 3 & 2 \\ 2 & -3 \\ 1 & 4 \end{matrix} \right|$$

Note: the Engr 170 textbook adds the first two rows below the original 3x3; this set-up also works.

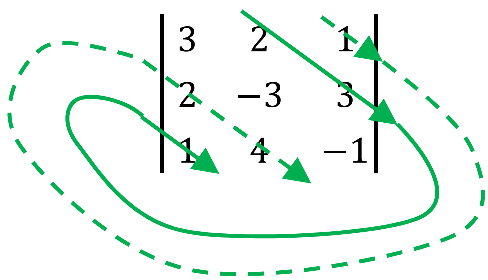
c. To calculate the determinant:

- Add the products of the 3 sets of 3 diagonal terms going down-right, and
- Subtract the products of the 3 sets of 3 diagonals going up-right.



$$= +(3)(-3)(-1) + (2)(3)(1) + (1)(2)(4) - (1)(-3)(1) - (4)(3)(3) - (-1)(2)(2)$$
$$= +(9) + (6) + (8) - (-3) - (36) - (-4)$$
$$= -6$$

If you do not write the first two columns to the right of the 3x3, when multiplying, just wrap around to the appropriate missing terms, e.g., the second and third positive products:



Step 2 Determinants Δ_1, Δ_2, …

Replace the first column of the coefficient matrix with the constant vector, and take the determinant of the matrix thus formed:

$$\Delta_1 = \det\begin{bmatrix} 4 & 2 & 1 \\ 6 & -3 & 3 \\ -5 & 4 & -1 \end{bmatrix} = \begin{vmatrix} 4 & 2 & 1 \\ 6 & -3 & 3 \\ -5 & 4 & -1 \end{vmatrix}$$
$$= +(4)(-3)(-1) + (2)(3)(-5) + (1)(6)(4) - (-5)(-3)(1) - (4)(3)(4) - (-1)(6)(2)$$
$$= +(12) + (-30) + (24) - (15) - (48) - (-12)$$
$$= -45$$

Repeat by replacing the second column of the coefficient matrix, and then the third:

$$\Delta_2 = \begin{vmatrix} 3 & 4 & 1 \\ 2 & 5 & 3 \\ 1 & -5 & -1 \end{vmatrix} = 31 \quad \text{and} \quad \Delta_3 = \begin{vmatrix} 3 & 2 & 4 \\ 2 & -3 & 6 \\ 1 & 4 & -5 \end{vmatrix} = 49$$

Step 3 Solving each Variable

The first variable is solved by dividing Δ_1 by Δ:

$$x = \frac{\Delta_1}{\Delta} = \frac{-45}{-6} = 7.50$$

Likewise:

$$y = \frac{\Delta_2}{\Delta} = \frac{31}{-6} = -5.17 \quad \text{and} \quad z = \frac{\Delta_3}{\Delta} = \frac{49}{-6} = -8.17$$

Alternate Method of Taking the Determinant of a 3x3 Matrix

An alternate method of taking the determinant of a 3x3 is to to break down the 3x3 matrix into three 2x2 matrices, as follows.

First, set up this matrix on the paper (or in your mind):

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

"+" signs are in positions where the Row + Column is even (1,1), (1,3), (2,2), (3,3), etc.

"–" signs are in positions where the Row + Column is odd (1,2), (2,3), etc.

Now, consider the elements in Row 1 of the 3x3. These will become coefficients of smaller 2x2 matrices as follows:

$$\Delta = \begin{vmatrix} \mathbf{3} & \mathbf{2} & \mathbf{1} \\ 2 & -3 & 3 \\ 1 & 4 & -1 \end{vmatrix} = +(3)\begin{vmatrix} -3 & 3 \\ 4 & -1 \end{vmatrix} - (2)\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + (1)\begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix}$$

The determinant of each 2x2 is found by adding the product of the downward diagonal terms and subtracting the product of the upward diagonal terms:

$$= +(3)[(-3)(-1) - (4)(3)] - (2)[(-2) - (3)] + (1)[(8) - (-3)]$$

$$= -27 + 10 + 11 = -6$$

Here is how the three 2x2's and their coefficients were set up.

- Take the element in Row 1, Column 1 (1,1): "3"; it will be multiplied by *positive* 1, the sign in (1,1) of the "+/–" matrix. Cover all elements in Row 1 and all elements in Column 1, which leaves four elements uncovered … a 2x2 matrix. The "+(3)" is multiplied by this 2x2 matrix.

$$\begin{vmatrix} 3 & 2 & 1 \\ 2 & -3 & 3 \\ 1 & 4 & -1 \end{vmatrix} \longrightarrow +(3)\begin{vmatrix} -3 & 3 \\ 4 & -1 \end{vmatrix}$$

- Take the element in (1,2): "2"; it will be multiplied by *negative* 1, the sign in (1,2) of the "+/–" matrix. Cover all elements in Row 1 and all elements in Column 2, which leaves four elements uncovered … a 2x2 matrix. The "–(2)" is multiplied by this 2x2 matrix.

$$\begin{vmatrix} 3 & 2 & 1 \\ 2 & -3 & 3 \\ 1 & 4 & -1 \end{vmatrix} \longrightarrow -(2)\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}$$

- Take the element in (1,3): "1"; a *positive* will be placed in front, the sign in (1,3) of the "+/–" matrix. Cover all elements in Row 1 and all elements in Column 3, which leaves four elements uncovered … a 2x2 matrix. The "+(1)" is multiplied by this 2x2 matrix.

$$\begin{vmatrix} 3 & 2 & 1 \\ 2 & -3 & 3 \\ 1 & 4 & -1 \end{vmatrix} \longrightarrow +(1)\begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix}$$

You can take any row or column you wish. Consider Column 2:

- Take the element in (1,2): "2"; a *negative* will be placed in front, the sign in (1,2) of the "+/–" matrix. Cover all elements in Row 1 and all elements in Column 2, which leaves four elements uncovered … a 2x2 matrix. The "–(2)" is multiplied by this 2x2 matrix.

$$\begin{vmatrix} 3 & 2 & 1 \\ 2 & -3 & 3 \\ 1 & 4 & -1 \end{vmatrix}$$

- Take the element in (2,2): "–3"; a *positive* will be placed in front, the sign in (2,2) of the "+/–" matrix. Cover all elements in Row 2 and all elements in Column 2, which leaves four elements uncovered … a 2x2 matrix. The "+(–3)" is multiplied by this 2x2 matrix.

$$\begin{vmatrix} 3 & 2 & 1 \\ 2 & -3 & 3 \\ 1 & 4 & -1 \end{vmatrix}$$

- Take the element in (3,2): "1"; a *negative* will be placed in front, the sign in (3,2) of the "+/–" matrix. Cover all elements in Row 3 and all elements in Column 2, which leaves four elements uncovered … a 2x2 matrix. The "–(4)" is multiplied by this 2x2 matrix.

$$\begin{vmatrix} 3 & 2 & 1 \\ 2 & -3 & 3 \\ 1 & 4 & -1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 3 & \mathbf{2} & 1 \\ 2 & \mathbf{-3} & 3 \\ 1 & \mathbf{4} & -1 \end{vmatrix} = -(2)\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} - (4)\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= -(2)[(-2) - (3)] + (-3)[(-3) - (1)] - (4)[(9) — (2)]$$

$$= 10 + 12 - 28 = -6$$

This method can be used to reduce higher-order matrices. For example, the determinant of a [6x6] matrix is found by:

- reducing the [6x6] to six [5x5]'s,
- reducing each [5x5] to five [4x4]'s ...30 total matrices
- reducing each [4x4] to four [3x3]'s …120 total matrices
- reducing each [3x3] to three [2x2]'s …360 total matrices

Take the determinant of 360 [2x2's] and multiply by appropriate products, making sure the sign works.

Or, you could just use MATLAB: define the 6x6 matrix, call it `A`, and then type `det(A)`.