Cross product of Two Vectors (Vector Algebra Method)

The cross product of two vectors is often used in engineering. In statics, the moment of force $\underline{\mathbf{F}}$ (a vector) about Point O is determined by taking the ***cross product***.

In three dimensions, consider vector $\underline{\mathbf{r}} = r_x\hat{\imath} + r_y\hat{\jmath} + r_z\hat{k}$, where $\underline{\mathbf{r}}$ is a position vector from Point O to any point on the *line of action* of vector force $\underline{\mathbf{F}} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ (Point A at the tail of $\underline{\mathbf{F}}$ is often a convenient point (Fig. 2-22)). Note that the scalar components may be positive or negative.

The moment $\underline{\mathbf{M}_O}$ about Point O is given by the cross product:

$$\underline{\mathbf{M}_O} = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$$

The order of the vectors in the cross product is important, and for the moment of a force, it is always "$\underline{\mathbf{r}}$ cross $\underline{\mathbf{F}}$". The result is a third vector, perpendicular to the plane formed by $\underline{\mathbf{r}}$ and $\underline{\mathbf{F}}$.

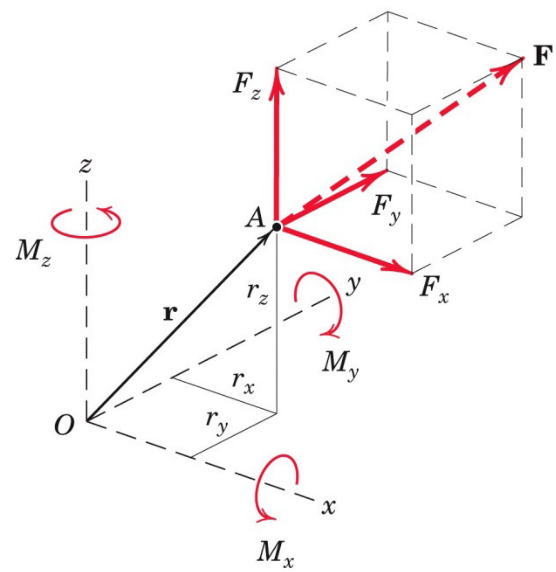

Fig. 2-22, Meriam and Kraige, *Engineering Mechanics: Statics*, 7e.

The cross product can be founding using vector algebra:

$$\underline{\mathbf{M}_O} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} = \left(r_x\hat{\imath} + r_y\hat{\jmath} + r_z\hat{k}\right) \times \left(F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}\right)$$

$$= r_xF_x(\hat{\imath} \times \hat{\imath}) + r_xF_y(\hat{\imath} \times \hat{\jmath}) + r_xF_z(\hat{\imath} \times \hat{k})$$

$$+ r_yF_x(\hat{\jmath} \times \hat{\imath}) + r_yF_y(\hat{\jmath} \times \hat{\jmath}) + r_yF_z(\hat{\jmath} \times \hat{k})$$

$$+ r_zF_x(\hat{k} \times \hat{\imath}) + r_zF_y(\hat{k} \times \hat{\jmath}) + r_zF_z(\hat{k} \times \hat{k})$$

Recall:

$$\hat{\imath} \times \hat{\jmath} = +\hat{k}\ ,\ \hat{\jmath} \times \hat{k} = +\hat{\imath}\ ,\ \hat{k} \times \hat{\imath} = +\hat{\jmath}$$

$$\hat{\jmath} \times \hat{\imath} = -\hat{k}\ ,\ \hat{k} \times \hat{\jmath} = -\hat{\imath}\ ,\ \hat{\imath} \times \hat{k} = -\hat{\jmath}$$

$$\hat{\imath} \times \hat{\imath} = 0\ ,\ \hat{\jmath} \times \hat{\jmath} = 0\ ,\ \hat{k} \times \hat{k} = 0$$

So:

$$\underline{\mathbf{M}_O} = r_xF_x(0) + r_xF_y(\hat{k}) + r_xF_z(-\hat{\jmath}) + r_yF_x(-\hat{k}) + r_yF_y(0) + r_yF_z(\hat{\imath}) + r_zF_x(\hat{\jmath}) + r_zF_y(-\hat{\imath}) + r_zF_z(0)$$

$$= \underbrace{\left(r_yF_z - r_zF_y\right)}_{M_{O,x}}(\hat{\imath}) + \underbrace{\left(r_zF_x - r_xF_z\right)}_{M_{O,y}}(\hat{\jmath}) + \underbrace{\left(r_xF_y - r_yF_x\right)}_{M_{O,z}}(\hat{k}) = M_{O,x}\hat{\imath} + M_{O,y}\hat{\jmath} + M_{O,z}\hat{k}$$

The scalar components of the moments are:

$M_{O,x} = r_yF_z - r_zF_y$ the moment about Point O about the x-axis.

$M_{O,y} = r_zF_x - r_xF_z$ the moment about Point O about the y-axis.

$M_{O,z} = r_xF_y - r_yF_x$ the moment about Point O about the z-axis.

These equations may seem hard to remember, but notice the pattern: (1) the moment about a specific axis is made up the r- and F-components from the other directions (e.g., $M_{o,y}$ only has r_x, r_z, F_x and F_z); (2) r always proceeds F, and (3) if the product's subscripts are in alphabetical order (x,y,z,x…), the product has a positive sign; if not, the product has a negative sign, e.g., : $+r_yF_z$, $+r_zF_x$ (z is at the end of the "alphabet", so the next letter is x), $-r_zF_y$, $-r_xF_z$.

Alternatively, the cross product of vector **r** with **F** can be found by calculating the ***determinant*** of the following matrix:

$$\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{bmatrix}$$

where the directional unit vectors are in the first row, the scalar components (positive or negative) of **r** are in the second row, and the scalar components of **F** are in the third row.

$$\underline{\mathbf{M}_O} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{bmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

To take the determinant, set up this matrix on the paper (or in your mind):

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

"+" signs are in positions where the Row + Column is even (1,1), (1,3), (2,2), (3,3), etc.

"–" signs are in positions where the Row + Column is odd (1,2), (2,3), etc.

Now, consider the elements in Row 1 (the unit vectors). These will become the coefficients of smaller 2×2 matrices, with the first and third terms being positive, and the second being negative:

$$\underline{\mathbf{M}_O} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = +(\hat{\imath}) \begin{vmatrix} r_y & r_z \\ F_y & F_z \end{vmatrix} \underbrace{-(\hat{\jmath})}_{\text{this term is negative}} \begin{vmatrix} r_x & r_z \\ F_x & F_z \end{vmatrix} + (\hat{k}) \begin{vmatrix} r_x & r_y \\ F_x & F_y \end{vmatrix}$$

The determinant of each 2×2 may be found by adding the product of the downward diagonal terms and subtracting the product of the upward diagonal terms:

$$\underline{\mathbf{M}_O} = +(\hat{\imath})(r_y F_z - r_z\, F_y) - (\hat{\jmath})(r_x F_z - r_z\, F_x) + (\hat{k})(r_x F_z - r_z\, F_x)$$

To keep all coefficients positive, the r- and F- terms of the $\hat{\jmath}$ coefficient are reversed:

$$\underline{\mathbf{M}_O} = (r_y F_z - r_z\, F_y)(\hat{\imath}) + (r_z\, F_x - r_x F_z)(\hat{\jmath}) + (r_x F_z - r_z\, F_x)(\hat{k}) = M_{O,x}\hat{\imath} + M_{O,y}\hat{\jmath} + M_{O,z}\hat{k}$$

The scalar components of the moments are:

$$M_{O,x} = r_y F_z - r_z F_y$$

$$M_{O,y} = r_z F_x - r_x F_z$$

$$M_{O,z} = r_x F_y - r_y F_x$$

Here is how the three 2×2's and their coefficients were set up.

- Take the element in Row 1, Column 1 (1,1): "$\hat{\imath}$"; it will be multiplied by *positive* 1, the sign in (1,1) of the "+/–" matrix. Cover all elements in Row 1 and all elements in Column 1, which leaves four elements uncovered … a 2×2 matrix. The "+($\hat{\imath}$)" is multiplied by this 2×2 matrix.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \longrightarrow +(\hat{\imath})\begin{vmatrix} r_y & r_z \\ F_y & F_z \end{vmatrix}$$

- Take the element in (1,2): "$\hat{\jmath}$"; it will be multiplied by *negative* 1, the sign in (1,2) of the "+/–" matrix. Cover all elements in Row 1 and all elements in Column 2, which leaves four elements uncovered … a 2×2 matrix. The "–($\hat{\jmath}$)" is multiplied by this 2×2 matrix.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \longrightarrow -(\hat{\jmath})\begin{vmatrix} r_x & r_z \\ F_x & F_z \end{vmatrix}$$

- Take the element in (1,3): "$\hat{k}$"; a *positive* will be placed in front, the sign in (1,3) of the "+/–" matrix. Cover all elements in Row 1 and all elements in Column 3, which leaves four elements uncovered … a 2×2 matrix. The "+($\hat{k}$)" is multiplied by this 2×2 matrix.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \longrightarrow +(\hat{k})\begin{vmatrix} r_x & r_y \\ F_x & F_y \end{vmatrix}$$

Moment – Right-Hand Rule

The main point of this handout was to explain the cross product, with the specific application of the moment. The moment can also be determined using the right-hand rule.

The moment of force $\underline{\mathbf{F}}$ about Point *A* (Fig. 2-8(b)), is:

$$\underline{\mathbf{M}_A} = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$$

where $\underline{\mathbf{r}}$ is the position vector from Point A to any point on the line of action of $\underline{\mathbf{F}}$.

The value of the moment is:

$$M_A = rF\ \sin\alpha$$

where α is measured from the $\underline{\mathbf{r}}$-direction to the $\underline{\mathbf{F}}$-direction (a positive moment implies counterclockwise, a negative moment clockwise).

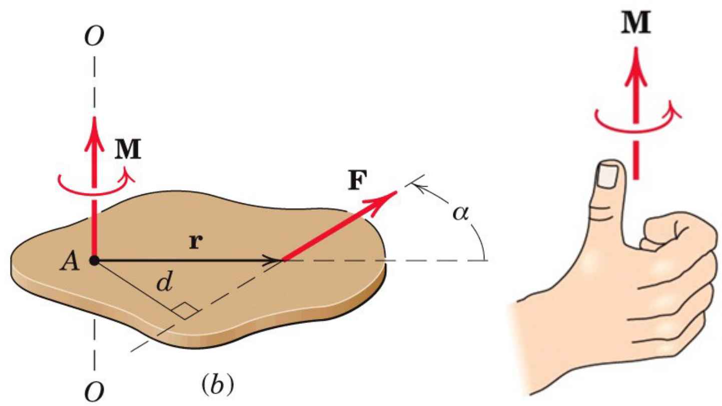

Fig. 2-8 (b) and (c), Meriam and Kraige, *Engineering Mechanics: Statics*, 7e.

The direction of $\underline{\mathbf{M}_A}$ is perpendicular the plane formed by $\underline{\mathbf{r}}$ and $\underline{\mathbf{F}}$, and found using the right-hand rule:

1. Point the fingers of your right hand in the direction of $\underline{\mathbf{r}}$.
2. Close your right-hand fingers into vector $\underline{\mathbf{F}}$.
3. The direction of your thumb is in the direction of $\underline{\mathbf{M}}$ (Fig. 2-8(c)). The direction your right fingers curl in is the direction of the action of the moment.

The moment vector is about the axis perpendicular to the plane that $\underline{\mathbf{r}}$ and $\underline{\mathbf{F}}$ both lie in.

Note that the magnitude of the moment can also be written:

$$M_A = |rF\ \sin\alpha| = |F(r\ \sin\alpha)| = Fd = |r(F\ \sin\alpha)| = rF_{\perp}$$

where d is the shortest distance between Point *A* and the line of action of $\underline{\mathbf{F}}$ (the perpendicular distance, or *moment arm*), and $F_{\perp}$ is the component of $\underline{\mathbf{F}}$ perpendicular to $\underline{\mathbf{r}}$. The magnitude of the moment can be thought of as the entire magnitude of F multiplied by the moment arm d, or the entire length r multiplied by the part of $\underline{\mathbf{F}}$ perpendicular to $\underline{\mathbf{r}}$.