Module #5 – Intro to Numerical Methods - Integration and Differentiation Engr 124 – Excel; F20	Name:

Instructions:

- Read the background sections (pp. 1-5).
  - Answer each problem on a different worksheet in **one workbook** (one Excel file).
  - Rename each worksheet with a one or two-word title that is descriptive of the problem.
  - Save an electronic copy of your file for reference. Name the file: **Last_name_Mod_5.xlsx**, e.g., Hancock_Mod_5.xlsx.
  - When you have **completed all the problems in the module**, upload the Excel file (workbook) to the Canvas web site for ENGR 124.

Formatting

- In the first Column (**A**), and in the first 3 rows of each worksheet, enter your **Name** (in **A1**), the **Problem Title** (**A2**) and the Date completed (**A3**).
- Start all work below **Row 4**.
- Make sure that you format each worksheet and use appropriate text (titles, prompts, etc.) so:
  (1) someone who opens the worksheet knows what the worksheet does, and
  (2) the user can easily use the sheet.
  (3) It looks good (background fill, borders, font).

1.0 INTEGRATION

The ***integral* of a function is the area under the curve generated by the function** (*Figure 1*). In calculus, the integral can be determined by dividing the area under the curve into infinitesimal areas, dA; e.g., vertical strips dx wide by $f(x)$ tall, as in *Figure 1*. By adding – integrating – all of these infinitesimal areas, the total area under the curve is determined:

$$I = \int dA = \int f(x)\,dx \qquad (1)$$

In Calculus 2, you are exposed to several techniques that allow you to ***integrate analytically*** (i.e., using variables). In other words, the result of the indefinite integral is a function; this is called a *closed formed solution*.

Perhaps because integration techniques can be complex, it is often forgotten that the ***integral is just the area of under the function***. Thus, if you can visualize the function, you can *estimate* its area (integral).

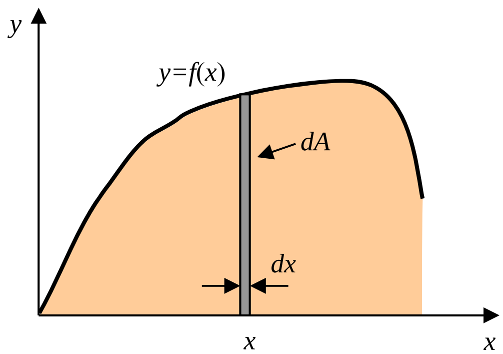

Figure 1 *The integral of f(x),* $\boldsymbol{I = \int f(x)dx}$ *, is the area under the f(x)-curve.*

2.0 NUMERICAL INTEGRATION

Many functions can be integrated analytically using the techniques of classical calculus. Other functions are either very difficult to evaluate analytically, or cannot be integrated analytically at all. The Gaussian function:

$$f(x) = e^{-x^2} \qquad (2)$$

cannot be integrated by the basic techniques that you would learn in a standard calculus course.

In many science and engineering applications, ***discrete*** (point-by-point) data cannot be described by a "nice" function. As you drive your car, if you plot speed v versus time t, the probability that your $v(t)$ vs. t curve will be a recognizable smooth function (or even several functions pieced together) is very small. **Real data is rarely described exactly by a nice function**; integration is problematic.

When $f(x)$ cannot be integrated analytically (or it is difficult to integrate), ***numerical integration*** is employed. The approach is the same as that of calculus – in fact, it is the basis of calculus. The total area A is first divided into small areas, A_i (i is a counting variable); the A_i's are then summed (*Figures 2–4*).

In *analytical integration*, the strips have infinitesimal width dx and finite height $f(x)$, and thus infinitesimal area $dA = f(x)dx$. In *numerical integration*, the strips have *finite width* Δx_i, and finite height $f(x_i)$. Thus, each area $A_i = f(x_i)\,\Delta x_i$ is *finite* (A_i has a value). The value used for $f(x_i)$ for each A_i depends on the method used (discussed below).

2.1 LEFT-RECTANGULAR METHOD

Consider *Figure 2.* The area under the $f(x)$ curve has been broken up into n rectangles each Δx_i wide; here $n = 8$. The width of each rectangle Δx_i can vary, although here they are drawn with the same width. A set of $(n+1)$ points on $f(x)$ are defined by the left and right x-values of each area: (x_o, y_o), (x_1, y_1),…(x_n, y_n). The height of each rectangle is determined by the value of $f(x)$ at the **left side of the rectangle**; thus the name of the method.

The finite area under the first rectangle is:

$$A_1 = (y_o)(x_1 - x_o) = y_o \Delta x_1$$

The area under the second rectangle is:

$$A_2 = (y_1)(x_2 - x_1) = y_1 \Delta x_2$$

The area under the ith rectangle is:

$$A_i = (y_{i-1})(x_i - x_{i-1}) = y_{i-1} \Delta x_i$$

The total area under the curve, I (for integral), is approximated by:

$$I \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} y_{i-1} \Delta x_i \qquad (3a)$$

where $\Delta x_i = x_i - x_{i-1}$. Again, interval Δx_i is generally not constant since data are not always equally spaced.

When $\Delta x_i = \Delta x$ is **constant** (all the intervals are the same width), the area under the curve reduces to the special solution:

$$I \approx \Delta x \sum_{i=1}^{n} y_{i-1} = \Delta x \left[y_o + y_1 + y_2 + \ldots + y_{n-1} \right] \qquad (3b)$$

Note that when $f(x)$ is increasing, the rectangles do not fill the entire area under the curve; when $f(x)$ is decreasing, the rectangles overestimate the area under the curve. Thus, the area calculated is only an approximation.

The smaller Δx, the more accurate the solution will be (when Δx goes to infinitesimal dx, we have an "exact" solution). Making Δx smaller requires more calculations, but the computer can be set up to do all the work in a short amount of time; that's what computers were made to do.

When the interval Δx_i is not constant, then you must do a little more work to set up the problem, but not much more, since the finite area is simply $A_i = y_{i-1}(x_i - x_{i-1})$, a calculation that can be easily automated.

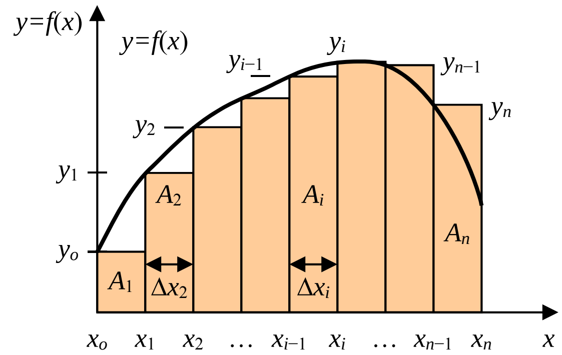

Figure 2 Left-Rectangular Method.

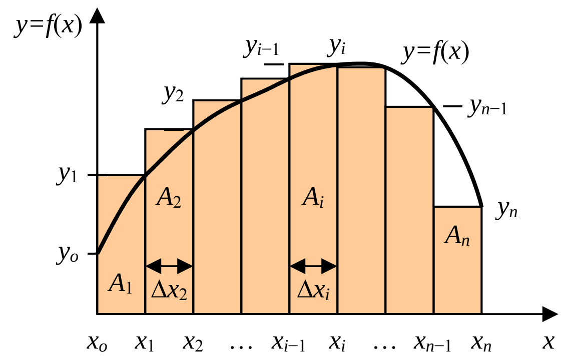

Figure 3 Right-Rectangular Method.

2.2 RIGHT-RECTANGULAR METHOD

Consider *Figure 3.* The area under the $f(x)$ curve has been broken up into n rectangles each Δx_i wide; here $n = 8$. The width of each rectangle Δx_i can vary. A set of $(n+1)$ points on $f(x)$ are defined by the left and right x-values of each area: (x_o, y_o), (x_1, y_1),…(x_n, y_n). The height of each rectangle is determined by the value of $f(x)$ at the **right side of the rectangle**; thus the name of the method.

Here, the area under the first rectangle is:

$$A_1 = (y_1)(x_1 - x_o) = y_1 \Delta x_1$$

The area under the second rectangle is:

$$A_2 = (y_2)(x_2 - x_1) = y_2 \Delta x_2$$

The area under the ith rectangle is:

$$A_i = (y_i)(x_i - x_{i-1}) = y_i \Delta x_i$$

The total area under the curve is:

$$I \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} y_i \Delta x_i \qquad (4a)$$

where $\Delta x_i = x_i - x_{i-1}$. Again, the interval Δx is generally not constant.

When Δx is constant, then:

$$I \approx \Delta x \sum_{i=1}^{n} y_{i-1} = \Delta x\left[y_1 + y_2 + \ldots + y_{n-1} + y_n\right] \qquad (4b)$$

Since some rectangles underestimate the area, and some overestimate it, the area calculated is thus only an approximation. The smaller Δx, the more accurate the solution will be.

As in the previous method, when the interval Δx_i is not constant, then you must do a little more work to set up the problem. However, the area is simply $A_i = y_i\left(x_i - x_{i-1}\right)$, and the computer can do all the work.

Note that for any set of data points, $n-1$ of the n A_i's of the Left-Rectangular Method and of the Right-Rectangular Method are the same. In general A_{i+1} of the Left-Rectangular Method equals A_i of the Right-Rectangular Method (e.g., $A_{2,left}=A_{1,right}$), as seen in Figures 2 and 3. The numerical difference of the two approximations is the difference in areas between A_1 of the Left-Rectangular Method, and A_n of the Right-Rectangular Method.

$$I_L = \Delta x y_o + \Delta x[y_1 + y_2 + \cdots + y_{n-1}]$$
$$I_R = \Delta x y_n + \Delta x[y_1 + y_2 + \cdots + y_{n-1}]$$

2.3 TRAPEZOIDAL METHOD

Rectangles of finite width are not the best shape to estimate the area under a curve. A better estimate is made using trapezoids, as illustrated in *Figure 4*. The area of a trapezoid is its average height multiplied by its width.

The area under the first trapezoid is:

$$A_1 = \left(\frac{y_o + y_1}{2}\right)\left(x_1 - x_o\right) = \left(\frac{y_o + y_1}{2}\right)\Delta x_1$$

The area under the second trapezoid is:

$$A_2 = \left(\frac{y_1 + y_2}{2}\right)\left(x_2 - x_1\right) = \left(\frac{y_1 + y_2}{2}\right)\Delta x_2$$

The area under the ith trapezoid is:

$$A_i = \left(\frac{y_{i-1} + y_i}{2}\right)\left(x_i - x_{i-1}\right) = \left(\frac{y_{i-1} + y_i}{2}\right)\Delta x_i$$

The total area under the curve is therefore:

$$I \approx \sum_{i=1}^{n} A_i = \frac{1}{2}\sum_{i=1}^{n}\left(y_{i-1} + y_i\right)\Delta x_i \qquad (5a)$$

where $\Delta x_i = x_i - x_{i-1}$. Again the interval Δx is generally not constant.

When Δx is constant, *Eq.* (5a) reduces to:

$$I \approx \Delta x \sum_{i=1}^{n}\left(\frac{y_{i-1} + y_i}{2}\right) = \frac{\Delta x}{2}\sum_{i=1}^{n}\left(y_{i-1} + y_i\right)$$
$$= \frac{\Delta x}{2}\left[y_o + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right]$$
$$= \frac{\Delta x}{2}\left[y_o + y_n + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right] \qquad (5b)$$

The Trapezoidal Rule is just the average of the Left and Right-Rectangular Methods.

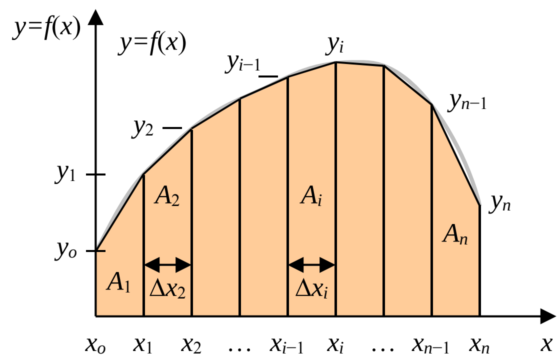

Figure 4 Trapezoidal Method.

2.4 SIMPSON'S RULE

Even better estimates can be made by fitting a parabola (quadratic) curve through three adjacent data point. These points are ideally equally spaced, distance Δx apart. Without derivation, ***Simpson's 1/3 Rule*** gives the area under such a curve:

$$I \approx \frac{1}{3}\sum_{i=1,3,5\ldots}^{n}\left(y_{i-1} + 4y_i + y_{i+1}\right)\Delta x \qquad (6)$$

Simpson's 1/3 Rule, *Simpson's 3/8 Rule*, and other advanced numerical integration techniques, will not be covered in this course.

3.0 DIFFERENTIATION

The ***derivative*** of a function is the slope of the curve at a particular location x:

$$f'(x) = \frac{df}{dx} = \text{slope of } f(x) \text{ at } x \qquad (7)$$

4.0 NUMERICAL DIFFERENTIATION –

FORWARD DIFFERENCE METHOD, BACKWARD DIFFERENCE METHOD, AND CENTRAL DIFFERENCE METHOD

For a set of data points, an approximation for the derivative at any point is the slope of a straight line between two points near the point of interest x:

$$f'(x) \approx \frac{\Delta f}{\Delta x} = \frac{\Delta y}{\Delta x} \qquad (8)$$

where Δy and Δx are the change in y- and x-values between the two points. The derivative is the ratio of the *difference* in the y-values to the *difference* in x-values between two points.

Consider *Figure 5*. The data points on the curve are: (x_o, y_o), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), … (x_n, y_n).

The slope at any point (x_i, y_i) can be approximated using the ***Forward Difference Method***. This method gives the slope of the line between (x_i, y_i) and the next point "forward" (x_{i+1}, y_{i+1}). For example, the Forward Difference Method approximates the slope at x_3 as:

$$f'(x_3) \approx \frac{(y_4 - y_3)}{(x_4 - x_3)} = \frac{(y_4 - y_3)}{\Delta x_4}$$

Or, the slope at (x_i, y_i) can be approximated using the ***Backward Difference Method***. This method gives the slope of the line between (x_i, y_i) and the next point "backward"(x_{i-1}, y_{i-1}). The slope at x_3 is then:

$$f'(x_3) \approx \frac{(y_3 - y_2)}{(x_3 - x_2)} = \frac{(y_3 - y_2)}{\Delta x_3}$$

The ***Central Difference Method*** gives a better approximation than the previous two methods (*Figure 5*). The Central Difference Method slope is the *average* of the slopes of the *Forward* and *Backward Difference Methods*. The slope at x_3 is:

$$f'(x_3) \approx \frac{1}{2}\left[\frac{(y_4 - y_3)}{\Delta x_4} + \frac{(y_3 - y_2)}{\Delta x_3}\right]$$

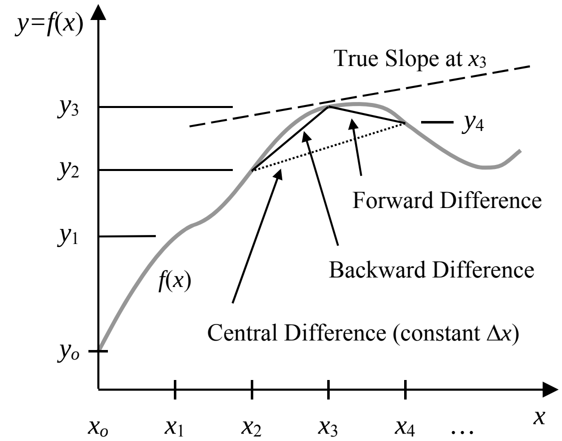

Figure 5 Estimating the slope at x_3 using the three difference methods. A smaller Δx would increase the accuracy of the methods.

Assuming Δx_i is constant, then the Central Difference approximation for the slope at x_3 is the average slope between (x_2, y_2) and (x_4, y_4):

$$f'(x_3) \approx \frac{(y_4 - y_2)}{2\Delta x}$$

Equations 9a 10a, and *11a* give the general expressions for the slope x_i for each method. The expressions in *Equations 9b*, *10b* and 11b, are only for the case when Δx is constant.

Forward Difference:

$$f'(x_i) \approx \frac{(y_{i+1} - y_i)}{(x_{i+1} - x_i)} \qquad (9a)$$

$$f'(x_i) \approx \frac{(y_{i+1} - y_i)}{\Delta x} \qquad (9b)$$

Backward Difference:

$$f'(x_i) \approx \frac{(y_i - y_{i-1})}{(x_i - x_{i-1})} \qquad (10a)$$

$$f'(x_i) \approx \frac{(y_i - y_{i-1})}{\Delta x} \qquad (10b)$$

Central Difference

$$f'(x_i) \approx \frac{1}{2}\left[\frac{(y_{i+1} - y_i)}{\Delta x_{i+1}} + \frac{(y_i - y_{i-1})}{\Delta x_i}\right] \qquad (11a)$$

$$f'(x_i) \approx \frac{(y_{i+1} - y_{i-1})}{2\Delta x} \qquad (11b)$$

As Δx decreases, the approximation for the slope improves. In the limit, Δx goes to dx.

4.1 Numerical Differentiation – The Second Derivative

The second derivative is the rate of change of the slope. Using the slopes from the Forward and Backward Difference Methods (*Equations 9* and *10*), and taking Δx as constant, the second derivative at x_i is approximated by:

$$
\begin{aligned}
f''(x_i) &= \frac{d^2y}{dx^2} \approx \frac{\Delta f'}{\Delta x} \\
&= \frac{f'(x_i)_{\text{forward}} - f'(x_i)_{\text{backward}}}{\Delta x} \\
&= \frac{\left[(y_{i+1} - y_i)/\Delta x\right] - \left[(y_i - y_{i-1})/\Delta x\right]}{\Delta x}
\end{aligned}
$$

Simplifying:

$$f''(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} \qquad (12)$$

Equation 12 is the ***Central Difference second derivative***.

Not surprisingly, there is a ***Forward Difference second derivative*** (just increase the subscripts by one):

$$f''(x_i) \approx \frac{y_{i+2} - 2y_{i+1} + y_i}{\Delta x^2} \qquad (13)$$

and a ***Backward Difference second derivative*** (just decrease the subscripts by 1):

$$f''(x_i) \approx \frac{y_i - 2y_{i-1} + y_{i-2}}{\Delta x^2} \qquad (14)$$

The ***Forward*** and ***Backward Difference*** second derivatives are not as accurate as the ***Central Difference*** second derivative. Use the ***Central Difference Method*** whenever possible.

4.2 Numerical Differentiation – End Points

Note that each differentiation method requires a data point *before* and/or *after* the point of interest. This requirement becomes a problem at the end points (x_o and x_n); there is no data point to the left of x_o, and no data point to the right of x_n . At the end points, only one of the three methods works.

At the leftmost data point, the ***Forward Difference Methods*** must be used to estimate slopes and second derivatives. At the rightmost data point, the ***Backward Difference Methods*** must be used.

For all interior points (non-end points, x_1 to x_{n-1}), the ***Central Difference Method*** should be used; it is the most accurate.

As a consequence of having to use the Forward Difference Method at the leftmost point, and the Backward Difference that the rightmost point, the calculated derivatives (first and second) at the endpoints are generally less accurate than the derivatives calculated at the rest of the points. However, the two endpoints typically represent a very small fraction of the overall data.

PROBLEMS for MODULE 5

NOTE: For hints on integrating with Excel, see the last two pages of this handout (two Excel worksheets).

Problem 1 *Integration*

Integrate $1/x$ from 1 to 2: $I = \int_1^2 \frac{dx}{x}$

Integrating by hand, to 7 digits: $I = \int_1^2 \frac{dx}{x} = \ln(x)\Big|_1^2 = \ln(2) - \ln(1) = \ln(2/1) = 0.6931472$

(a) Use the ***Left-Rectangular Method*** and Excel to integrate numerically.

To do so, break the distance from 1 to 2 into 10 intervals of equal width, i.e., 11 points (x_0=1.0, x_1=1.1, x_2=1.2… x_{10}=2.0), which are the sides of 10 rectangles. Note the first data point is (x_0, y_0).

Calculate the area of each rectangle individually, $A_i = y_{i-1}\Delta x$, and then add all 10 areas A_i together. Since this is the Left-Rectangular Method, the heights of the rectangles are: y_0, y_1, y_2… y_9. The first area is $A_1 = y_0\Delta x$.

See Figure 6 for sample table set-up.

(b) Use the ***Right-Rectangular Method*** and Excel to integrate numerically.
Break the distance from 1 to 2 into 10 intervals of equal width (i.e., use 10 rectangles).
Calculate the area of each rectangle individually, $A_i = y_i\Delta x$, and then add all 10 areas A_i together. The first area is $A_1 = y_1\Delta x$.

(c) Use the ***Trapezoidal Method*** and Excel to integrate numerically.
Break the distance from 1 to 2 into 10 intervals of equal width (i.e., use 10 rectangles).
Calculate the area of each trapezoid individually, A_i , and then add all 10 areas together. The first area is $A_1 = [(y_0+y_1)\Delta x]/2$.

(d) Use Excel to calculate the percent error of each method with respect to the actual value of 0.6931472. Display the error below the calculation of each total area ΣA_i . The formula for the percentage errors is:

$$\%error = \frac{approximation - actual}{actual} \times 100\%$$

A positive error indicates an overestimation; a negative error indicates a underestimation.

A recommended format for *Problem 1* is shown in Figure 6.

Integration of 1/x from 1 to 2.					
Constant Δx = 0.1					
Data Point			**Left**	**Right**	**Trapezoid**
i	***x_i***	***y_i***	***A_i,***	***A_i,***	***A_i,***
0	1.0	`y0 calc.`			
1	1.1	`y1 calc.`	`A1 calc.`	`A1 calc.`	`A1 calc.`
2	1.2		`A2 calc.`	`A2 calc.`	`A2 calc.`
…	…	…	…	…	…
10	2.0	`y10 calc.`	`A10 calc.`	`A10 calc.`	`A10 calc.`
		Total Area:	`ΣA calc.`	`ΣA calc.`	`ΣA calc.`
		Error:	`Error calc.`	`Error calc.`	`Error calc.`

Figure 6 Sample format for *Problem 1*. Excel should calculate the values indicated by the typeface **`Courier New`** (e.g., `y0 calc., A1 calc.`).

Problem 2 *Integration – 20 intervals*

(a) Repeat Part (a) of Problem 1, but with 20 rectangles. In other words, use the ***Left-Rectangular Method***, and break the distance from 1 to 2 into **20 intervals** of equal width.

(b) Calculate and display the error in the 20-interval Left-Rectangular result.

(c) For the *Left-Rectangular Method*, compare the 20-interval solution (Prob. 2a) to the 10-interval solution (*Prob. 1a*). Indicate in the worksheet which solution is more accurate.

Problem 3 *Derivative and Integral*

Consider the following *x-y* data:

x	y
0	0
0.5	2.0
1.1	5.0
2.2	8.0
2.8	10.0
4.0	12.0
5.6	17.0
7.0	18.0
8.4	20.0
10.0	22.0

Note: The spacing Δx for this problem is **not constant**.

(a) Use the ***Forward Difference Method*** to approximate the slope at every point (except the last).

(b) Use the ***Backward Difference Method*** to approximate the slope at every point (except the first)

(c) Approximate the area under the curve formed by the *x-y* data using the ***Right-Rectangular Method***.

Problem 4 *Integral and Derivative:*
Given the Velocity-Time Data, Determine the *Distance Traveled* and the *Acceleration* over the Entire Time-Interval.

Note: For help in setting up the worksheet for this problem, see the next page: "PROBLEM 3: Velocity vs. Time Data" The time-step Δt is constant. Use the data shown.

Consider the *velocity-time* data in the spreadsheet on the following page.

Set up an Excel worksheet to solve for the object's *acceleration* and *position* with time. Then, plot (1) position vs. time, (2) velocity vs. time, and (3) acceleration vs. time. In particular:

(a) Create an Excel worksheet like that on the following page. Enter the velocity vs. time data shown.

(b) *Acceleration* is the derivative of velocity (the slope of the *v-t* curve). Calculate the acceleration of the object at all times using all three derivative methods – *Forward*, *Backward*, *Central* (note that at each endpoint, only one method can be used).

(c) *Change in position* is the integral of velocity (the area under the *v-t* curve). Calculate the area under the *v-t* curve using the *Trapezoidal Rule*. If the object starts at $x = 0$, then its *position* at time *t* (location) is also its *change in position*.

(d) Plot three separate graphs (*x-y* scatter plots) of:

1. *position* vs. *time*
2. *velocity* vs. *time*
3. *acceleration* vs. *time* (use the *Central Difference Method* results in general, and the *Forward* and *Backward Difference Methods* at the endpoints).

The graphs should be the same size, and horizontally aligned left. They should be arranged, top to bottom: position, velocity and acceleration.

Do your plots actually <u>look</u> like the derivatives / integrals of each other?

PROBLEM 3 **: Velocity vs. Time Data** Δt = constant. DJD 11/01/12

EXAMPLE EQUATIONS are shown in D6, D7, E7, F7, H7 and I7.

Data Point	Time (sec)	Velocity (m/s)	$\Delta v/\Delta t$, Forward Difference	$\Delta v/\Delta t$, Backward Difference	$\Delta v/\Delta t$, Central Difference	**Acceleration, *a* (m/s^2)** $a = \Delta v/\Delta t$, for plotting	A_i, Trapezoid	**Position, Δx (m)** Running Total of Area
0	0	5.0	=(C7-C6)/(B7-B6)					0.0
1	2	5.5	=(C8-C7)/(B8-B7)	=(C7-C6)/(B7-B6)	=(C8-C6)/(B8-B6)		=0.5*(C6+C7)*(B7-B6)	=H7+I6
2	4	6.2						
3	6	6.1						
4	8	6.5						
5	10	7.7						
6	12	9.5						
7	14	11.7						
8	16	13.6						
9	18	14.5						
10	20	14.5						
11	22	13.9						
12	24	11.5						
13	26	10.7						
14	28	9.9						
15	30	9.1						
16	32	8.3						
17	34	6.2						
18	36	5.7						
19	38	5.2						
20	40	5.0						

Notes:
1. Backward and Central Difference do not work at the first data point.
2. Forward and Central Difference do not work for the last data point.
3. Δt = constant, so Central Difference is a basic calculation.
   If Δt is not constant, the formula is an approximation of the averages of the Forward and Backward Differences.
4. The area, A_i, calculated by the Trapezoid rule is the area between data points
   data points x_i and x_{i-1}. e.g., $A_1 = 0.5*(y_1+y_0)*(x_1-x_0)$; $A_5 = 0.5*(y_5+y_4)*(x_5-x_4)$

***EXAMPLE*: Integrating the Gaussian Function**

DJD 11/01/12

Integrate **$f(x) = \exp(-x^2) = e^{-x \wedge 2}$**
from 0 to 1

Form of Equations in each Column.

Equation in:	Cell D11	Cell E11	Cell F11
	`=(C10)*(B11-B10)`	`=C11*(B11-B10)`	`=0.5*(C10+C11)*(B11-B10)`

Data Point	*x*-value	$f(x_i)$	A_i, Left Rectangular	A_i, Right Rectangular	A_i, Trapezoid
0	0.0	1			
1	0.1	0.990049834	0.1	0.099004983	0.099502492
2	0.2	0.960789439	0.099004983	0.096078944	0.097541964
3	0.3	0.913931185	0.096078944	0.091393119	0.093736031
4	0.4	0.852143789	0.091393119	0.085214379	0.088303749
5	0.5	0.778800783	0.085214379	0.077880078	0.081547229
6	0.6	0.697676326	0.077880078	0.069767633	0.073823855
7	0.7	0.612626394	0.069767633	0.061262639	0.065515136
8	0.8	0.527292424	0.061262639	0.052729242	0.056995941
9	0.9	0.444858066	0.052729242	0.044485807	0.048607525
10	1.0	0.367879441	0.044485807	0.036787944	0.040636875
		Total Area = Integral	0.777816824	0.714604768	0.746210796

Accepted Value of Integral to 4 places:	0.7468

Percent Error:	4.15	-4.31	-0.08