Module #3 - Solving Equations Engr 124 – Excel, F20	Name:

Objective   To:
- solve equations using a brute force method;
- use the *Goal Seek* tool to solve equations;
- use the *Solver* tool to solve systems of equations.

Instructions
- The PROBLEMS for Module 3 are on **Page 8**. The first few pages describe what is to be done. **You should practice the methods by actively following the examples on Pages 1-7.**
- Answer each problem on a different worksheet in <u>**one**</u> **workbook** (one Excel file).
- Rename each worksheet with a one or two-word title that is descriptive of the problem.
- Save an electronic copy of your file for reference. Name the file: **Last_name_Mod_3.xlsx**, e.g., Hancock_Mod_3.xlsx.
- When you have **completed all the problems in the module**, upload the Excel file (workbook) to the Canvas web site for ENGR 124.

Formatting
- In the first Column (**A**), and in the first 3 rows of each worksheet, enter your **Name** (in **A1**), the **Problem Title** (**A2**) and the Date completed (**A3**).
- Start all work below **Row 4**.
- Make sure that you format each worksheet and use appropriate text (titles, prompts, etc.) so:
  - (1) someone who opens the worksheet knows what the worksheet does, and
  - (2) the user can easily use the sheet.
  - (3) It looks good (background fill, borders, font).

Solving Equations

This module is meant to give you several mathematical tools to solve for the roots of equations, or even systems of equations. While Excel can deal with matrices, MATLAB is a much better tool for dealing with such mathematics. MATLAB is taught in Engr 126, offered in Spring.

Excel provides several ways to solve equations and systems of equations. Three methods are illustrated:
1. The Monkey Method (Brute Force Method).
2. *Goal Seek* (an Excel tool).
3. *Solver* (an Excel *Add-in* that must be activated to use).

1. *Monkey Method (Brute Force Method)*

What I call the *Monkey Method* uses the computer's power to perform many calculations rapidly. The approach is to calculate the value of a function for many inputs that bracket the solution until we get as close to the actual solution as desired.

By hand, this repetitive method would be lengthy and tedious, but by using the computer's brute force, the method becomes quick and elegant in its own way.

Consider the equation:

$$x^2 + 2\sin x = 4$$

where x is in radians. It is desired to solve for x.

To solve for x, rearrange the terms to create the mathematical function:

$$f(x) = x^2 + 2\sin x - 4$$

The solution $x = x_o$ is where $f(x) = 0$. In the Monkey (Brute Force) Method, just calculate $f(x)$ many times. Be we need to be clever about it.

An *initial guess* of where solution x_o lies must be made – select a *range* from x_{1i} to x_{1f}. Then break the range up into several increments and calculate the value of the $f(x)$ at the x-values defined by the increments.

Assuming the solution x_o lies between x_{1i} and x_{1f}, then x_o lies between the two values of x where $f(x)$ changes sign since $f(x)$ must go through zero between those two points. Let us call these two new x-values x_{2i} and x_{2f}. These two x-values are the endpoints of the (smaller) range for the second iteration of the method (thus the 2 in the subscript).

This "sifting" method if repeated until we have as many digits in the estimate of x_o as we desire (e.g., to two decimal places), or the required tolerance is reached (e.g., when the next the change in x_o will be less than 0.1% of the current estimate of x_o). Alternatively, we can stop when we get as close to $f(x) = 0$ as desired; e.g., $f(x) = 0.0001$ or 0.00001, etc.

CAUTION:
- If our initial range does not contain the solution, the range can be made larger (or shifted) until a $f(x) = 0$ is bracketed within the range.
- If a solution does not exist, we will never find $f(x) = 0$; e.g., the solution to the quadratic is complex.
- If more than one solution exists, we may not find all of them.

Back to the example:

$$f(x) = x^2 + 2\sin x - 4$$

Iteration 1. Start with a range that we guess the solution is in. Here select the range of $x = 0$ to 10 ($\sin x$ can have a magnitude no greater than 1, so $2\sin x$ cannot be greater than 2).

Now select an increment about 1/10 of the range, here 1.0 (*Figure 3.1*). When $f(x)$ changes sign, the solution x_o must lie between the two adjacent x-values that give $f(x)$ of opposite sign. Here, x_o must be between 1 and 2.

Iteration 2. Repeat Iteration 1 but with new limits 1.0 and 2.0, and an increment of 0.1. Function $f(x)$ changes sign between 1.4 and 1.5.

Iteration 3. Repeat with new limits 1.40 and 1.50, with an increment of 0.01. Function $f(x)$ changes sign between 1.42 and 1.43.

One more iteration gives $x_o = 1.422$ for the solution. Here, $x_o = 1.422$ is only one of two solutions.

	A	B	C	D	E	F	G	H
1	**Solution for:**							
2	$f(x) = x^2 + 2\sin x - 4$			Monkey Method (Brute Force Method)				
3								
4	Iteration 1: 0 to 10			Iteration 2: 1.0 to 2.0			Iteration 3: 1.40 to 1.50	
5		1 Δ			0.1 Δ			0.01 Δ	
6	**x**	**f(x)**		**x**	**f(x)**		**x**	**f(x)**
7	0	-4.000		1.0	-1.317		1.40	-0.0691
8	**1**	**-1.317**		1.1	-1.008		1.41	-0.0377
9	**2**	**1.819**		1.2	-0.696		**1.42**	**-0.0063**
10	3	5.282		1.3	-0.383		**1.43**	**0.0251**
11	4	10.486		**1.4**	**-0.069**		1.44	0.0565
12	5	19.082		**1.5**	**0.245**		1.45	0.0879
13	6	31.441		1.6	0.559		1.46	0.1193
14	7	46.314		1.7	0.873		1.47	0.1507
15	8	61.979		1.8	1.188		1.48	0.1822
16	9	77.824		1.9	1.503		1.49	0.2136
17	10	94.912		2.0	1.819		1.50	0.2450

Figure 3.1 Monkey Method or Brute Force Method.

2. *Goal Seek*

Goal Seek is a useful Excel tool that allows you to specify the **goal** or **target value** of one cell. The value of this **goal cell** depends on the value of another cell. When Goal Seek is executed, Excel will change the value of the other cell until the goal cell's target value is reached.

The **goal cell** (the *dependent cell*) must contain a **formula** that depends on the value of the other cell that is allowed to change (the *independent cell*). The independent cell must contain an actual *number*; it cannot contain a formula. This number is the *seed*, or first guess. Note that *dependent* and *independent* cells – as in dependent and independent variables – are my terms… I do not think they are used in Excel's Help.

The Goal Seek dialog box is accessed through:

DATA Tab/Data Tools/What-If Analysis/Goal Seek

Consider the equation:

$$x^2 - 5x + 6 = 0$$

To solve for x using Goal Seek, create the mathematical function:

$$f(x) = x^2 - 5x + 6$$

The solution x_o is found when $f(x) = 0$. We know the two solutions to be $x_o = 2$ and 3.

Figure 3.2 shows the Goal Seek dialog box opened, and hovering on top of the worksheet. **Cell A5** contains the value of x and **goal cell A6** contains $f(x)$. The *initial guess* for x – or **seed value** – is $x = 8$, so right now, $f(x) = 30$. An initial guess is necessary to tell Excel where to begin its solution method.

In the Goal Seek dialog box, the target **Cell A6** has been set to a *target value* of 0; this goal will be reached by Excel changing the *number* in **A5**. Excel will do this on its own, using a solution technique beyond the scope (and space) of this text.

Once the target value and cell addresses are specified, select $\boxed{\text{OK}}$.

Figure 3.3 shows the results of the numerical analysis. Excel will stop when it reaches a certain tolerance, and report that it has "found a solution."

Since this is a numerical solution, the result will not necessarily be the exact solution. In *Figure 3.3*, 5.31148E–05 (5.31148×10^{-5}) is zero, or within the

tolerance specified by the solution method. By formatting the cell to display only three digits after the decimal, the result in this case will be the more familiar $f(x) = 0.000$ when $x_o = 3.000$

CAUTION:
- Goal Seek will only find one solution. The solution found will depend on the initial value of the cell – the *initial guess* or *seed value* – and the form of the equation. If we started at a different value, we might have found the other solution, $x_o = 2$.
- If your goal does not exist, Excel will not find it; it will warn you that it "may not have found a solution" (*Figure 3.4*).
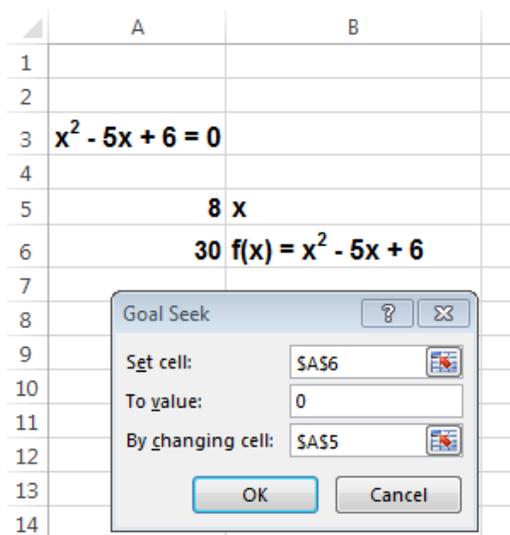
Figure 3.2. Goal Seek Dialog Box.
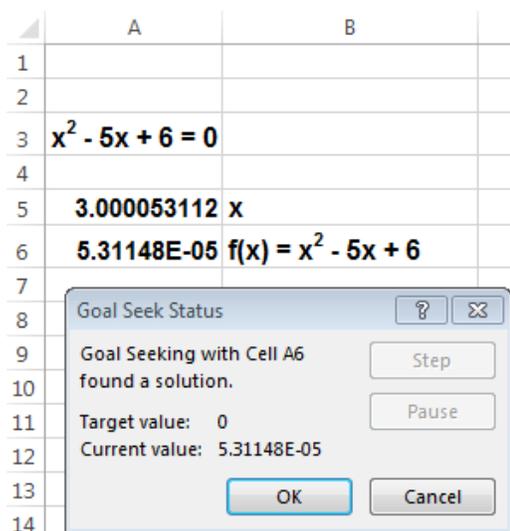
Figure 3.3. Results of Goal Seek.

Figure 3.4 shows the result of Goal Seek when no solution exists. Here, we have tried to find the zeroes of:

$$f(x) = x^2 + 4, \quad \text{i.e.,} \quad x^2 = -4$$

The solution is not real. Excel has continued to move along the x-axis (to $x = -335544.32$) trying to find the solution to $x^2 - 4 = 0$, but could not find it since the solution is not real ($x = \pm 2i$).
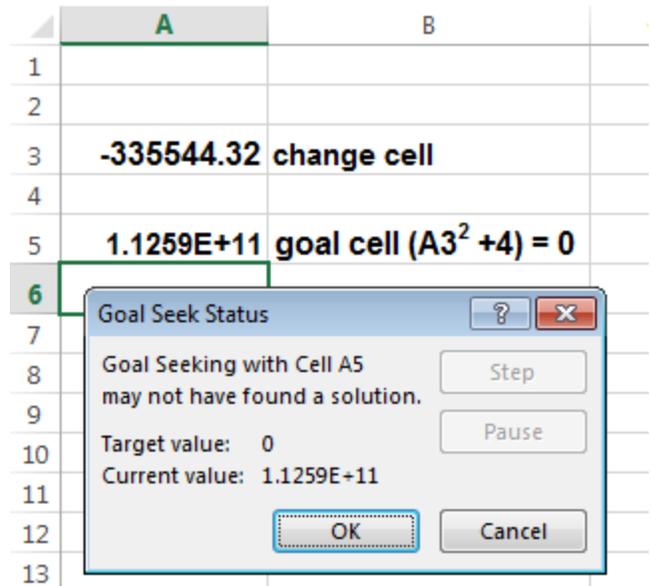
Figure 3.4. Result of setting goal of A5 = A3^2 = −4.

3. *Solver Tool*

The **Solver** tool allows you to solve more complicated problems than the Goal Seek tool.

The **Solver** is an **Add-in** that must be activated. If it does not appear in the **DATA Tab** in a group called **Analysis**, you must activate it.

Select **FILE Tab/Excel Options**.

The dialog box in *Figure 3.5* will appear. Select **Add-ins**, and at the **Manage** prompt near the bottom, select **Excel Add-ins/GO**.

The dialog box in *Figure 3.6* will open. Select **Solver Add-in** (and any other add-ins you want; I usually add the Analysis ToolPak which allows you to create histograms and use other data analysis tools).
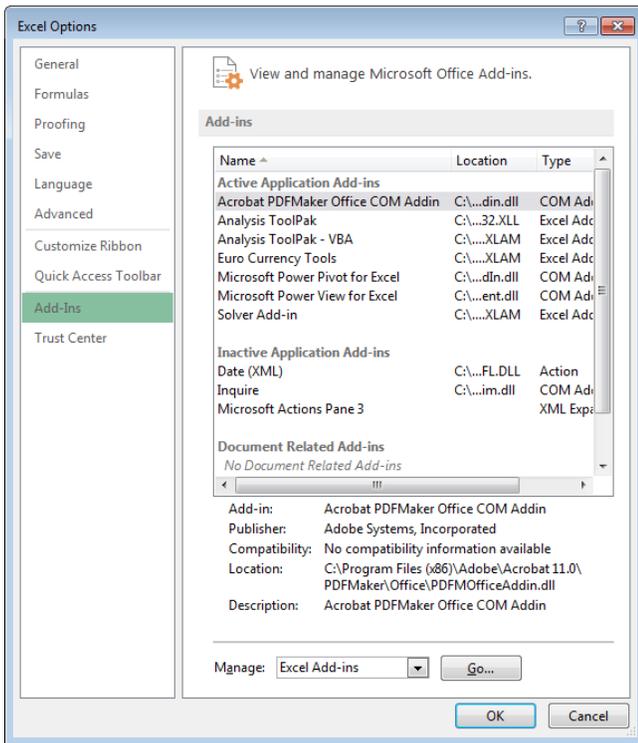
Figure 3.5. Excel Options.
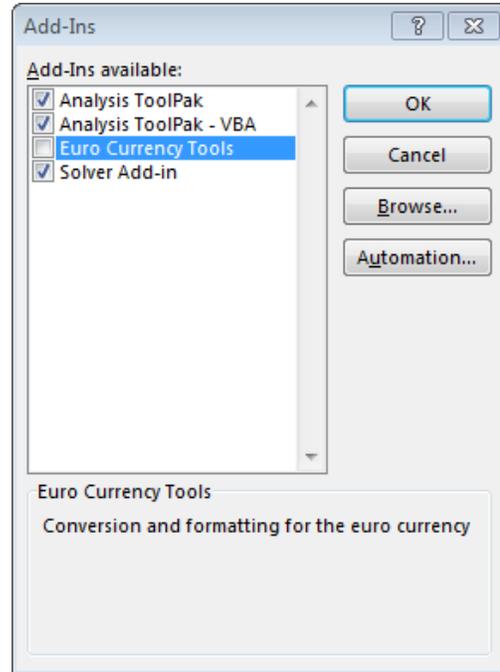
Figure 3.6. Add-Ins dialog box.

Solution Method

As an example, consider the following system of equations:

$$7x + 2y - 4z = 5$$
$$3x - 2y + 2z = 7$$
$$x + 7y - 5z = 11$$

We will solve this system with **Solver**. First, rearrange the equations to create three functions:

$$f(x,y,z) = 7x + 2y - 4z - 5$$
$$g(x,y,z) = 3x - 2y + 2z - 7$$
$$h(x,y,z) = x + 7y - 5z - 11$$

The solution is (x_o, y_o, z_o).

Like **Goal Seek**, we are going to change the input to get one *target output*. For **Solver**, however, the input may have more than one *independent cell*. Many inputs contributing to a single output is common in economics and business, in science and engineering, and in life in general.

For our system of equations, the inputs are x, y and z. The output would seem to be the values of functions f, g and h, which we have set up so the each will equal zero for solution x_o, y_o and z_o. However, **Solver** can have only *one* target cell; we cannot have three targets.

The TRICK

We need some clever way to use one target cell to ensure that all three functions (f, g and h) go to zero.

We might first consider that when each function goes to zero, then the **sum** of the functions must also go to zero. Unfortunately, while the sum is a *single output*, it does not guarantee we have the solution; e.g., the sum of $f = +6$, $g = +4$ and $h = -10$, would be "a solution."

The key (trick) is to notice that the **sum of the squares** of the functions is always non-negative:

$$\Delta(x,y,z) = f^2 + g^2 + h^2 \geq 0$$

regardless of their positive or negative sense. Thus, when the sum of the squares is zero, we must be at the solution (x_o, y_o, z_o) and $f = g = h = 0$. Thus:

$$\Delta(x_o, y_o, z_o) = 0$$

Pretty slick, huh? I wish I had thought of it.

Note that the solution (x_o, y_o, z_o) also minimizes Δ.

IMPLEMENTING the METHOD in EXCEL

Set up the spreadsheet shown in *Figure 3.7*.

Cells A2, **A3** and **A4** contain x, y and z, respectively (in fact, I actually renamed the cells **x**, **y** and **z** using *name references*). I have seeded these cells all with the value of 1.0 **Cells A7**, **A8** and **A9** contain functions f, g and h, respectively. **Cell A11** – the target cell – contains the formula for Δ.

Now, select: **Data Tab/Analysis/Solver.**

This opens the Solver dialog box (*Figure 3.8*), in which you will enter the Solver parameters. Here*,* **Cell A11** has a *target value* of 0, to be determined by changing cells in the range **A2:A4** (Excel has automatically added *absolute reference* indicators – the $-symbols).

The results of running solver are shown in *Figure 3.9*:

$$(x_o, y_o, z_o) = (2.21, 4.86, 5.05)$$

The value of Δ, 6.041×10^{-10}, is essentially zero, as are the values of f, g and h.

	A	B	C	D
1	Initial Values for Solver			
2	1	x		
3	1	y		
4	1	z		
5				
6				
7	0	$f(x,y,z) = 7x + 2y - 4z - 5 = 0$		
8	-4	$g(x,y,z) = 3x - 2y + 2z - 7 = 0$		
9	-8	$h(x,y,z) = x + 7y - 5z - 11 = 0$		
10				
11	80	sum of squares $= f^2 + g^2 + h^2$		

Fig. 3.7 Setting up for Solver.
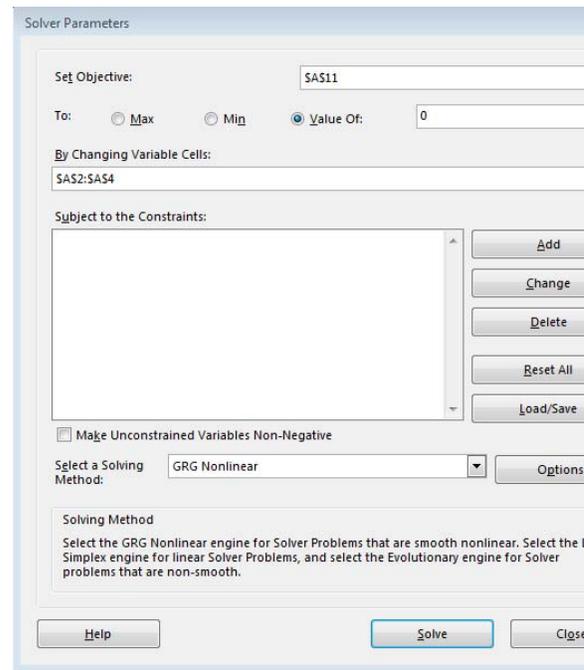
Fig. 3.8 Solver dialog box. The objective cell is $A$11 and its desired value is 0, which will be found by changes cells in range $A$2:$A$4.

CAUTION: Sometimes "Make Unconstrained Variables Non-Negative" is checked. This means your variables must be zero or positive, which makes sense if you are dealing with quantities that must be positive. If variables can be negative, uncheck this box.

	A	B	C	D
1	Initial Values for Solver			
2	2.2093045	x		
3	4.8604608	y		
4	5.046509	z		
5				
6				
7	1.666E-05	$f(x,y,z) = 7x + 2y - 4z - 5 = 0$		
8	9.873E-06	$g(x,y,z) = 3x - 2y + 2z - 7 = 0$		
9	-1.513E-05	$h(x,y,z) = x + 7y - 5z - 11 = 0$		
10				
11	6.041E-10	sum of squares $= f^2 + g^2 + h^2$		

Fig. 3.9 Solution.

CONSTRAINTS

Constraints can also be added to the solution in **Solver**. Constraints are limits on the values of *x, y* and *z*. Constraints are always an issue in real life… *x, y,* and *z* might represent the number of people to be assigned to certain projects, and *x* might have a minimum (or maximum) value, for example, you might need to send at least 3 people to a certain job.

Solver was run again for the same equations, this time with the constraint that *y* (**Cell A3**) must be less than or equal to 3.0 (*Figures 3.10* and *3.11*). We will not get "the solution" of $\Delta = 0$ – none of the functions *f, g* and *h* go to zero. However, we did get the solution that best minimizes the sum of the squares of the three functions, *f, g* and *h*. This is the optimal solution given the constraints. *Optimization* is an important field of study in engineering and economics.

OPTIONS

Solver Options (*Figures 3.12*) include the maximum time allotted for solution, the maximum number of numerical iterations, precision (how many digits to go to); tolerance (how close to the actual solution is acceptable), etc. The details of the numerical solution method (beyond the scope of this text), can also be set.
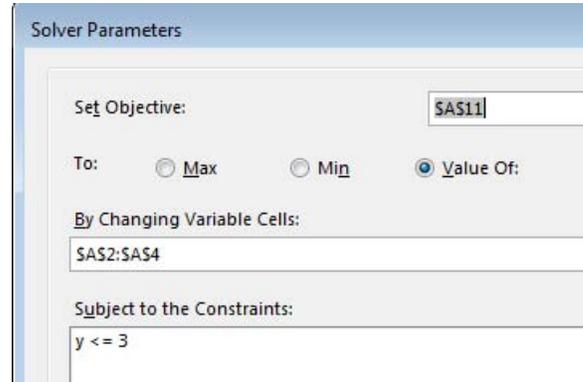
Fig. 3.10 Adding a constraint, *y* (Cell B5 was named *y*) must be no greater than 3.

	A	B	C	D
1	**Initial Values**	**for Solver**		
2	1.635514	x		
3	3	y		
4	2.7590862	z		
5				
6				
7	1.4122534	f(x,y,z) = 7x + 2y - 4z - 5 = 0		
8	-2.5752856	g(x,y,z) = 3x - 2y + 2z - 7 = 0		
9	-2.1599169	h(x,y,z) = x + 7y - 5z - 11 = 0		
10				
11	13.291796	sum of squares = $f^2 + g^2 + h^2$		

Fig. 3.11 Solution with constraint of *Fig. 3.9.*
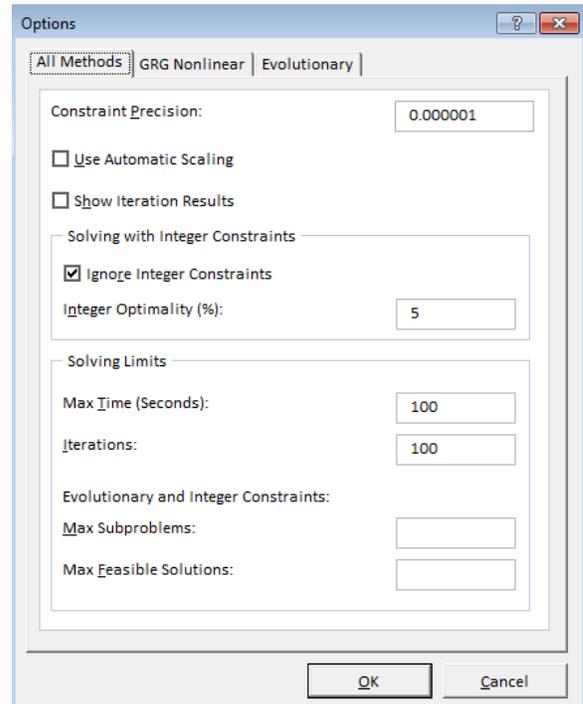
Fig. 3.12 Solver Options.

Module 3   PROBLEMS

PROBLEM 1 (save in Sheet1):

(a) $x^2 + 2\sin x = 4$ has one solution at $x = 1.422$ (x in radians)
Use the Monkey Method to find the other solution.
Give your answer to 3 digits after the decimal point.
Hint: expand/adjust the initial range. *Hint 2*: the two solutions both lie between –10 and 10.

(b) Using the Monkey Method, find two solution for: $7\sqrt{x} = 17 - 5\cos 2x$
such that $0 < x < 5.000$. Give your answer to 4 significant digits.

PROBLEM 2 (Sheet2):

Use **Goal Seek** to find the solution(s) to: $x^3 + 8x - 7 = 0$

PROBLEM 3: (Sheet3):

Use **Solver** to find the solution the following system of equations:

$$x + 2y + 3z + 4w = 5$$
$$x - 3y + 5z - 7w = 9$$
$$3x + 5y - 7z + 3w = 5$$
$$2x + 3y + 5z + 6w = -4$$

CHALLENGE PROBLEM 1: (Sheet4):

Set up a worksheet so that anyone who opens it can use it to solve a 3×3 system of equations with **Solver**.
The user knows how to open Solver, and can specify the independent cells and the target cell value.

You must create the <u>interface</u> for the 3×3 system. The user will enter the following into cells of your
choosing:

<u>User Inputs</u>:   The coefficients and the constants, *a,b,c...j, k l.*
Initial values of *x, y, z* (or you can include them in the sheet to being with.

$$ax + by + cz = j$$
$$dx + ey + fz = k \quad \text{or, as a matrix:} \quad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} j \\ k \\ l \end{bmatrix}$$
$$gx + hy + iz = l$$

CHALLENGE PROBLEM 2: (Sheet5):

Set up a worksheet so that anyone who opens it can use it to solve a 3×3 system of equations using determinants.

Cramer's Rule is another method use to solve systems of equations. In this method, determinates of matrices are calculated, and the solution found by taking ratio of various determinates.

For example, consider 3×3 system:

$$ax + by + cz = j$$
$$dx + ey + fz = k \quad \text{or, as a matrix:} \quad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} j \\ k \\ l \end{bmatrix}$$
$$gx + hy + iz = l$$

The determinant of the 3×3 coefficient matrix is:

$$\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb$$

To solve for x, replace the first column of the constant matrix with the constant vector and solve for that determinate

$$\Delta_x = \begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix} = jei + kfg + ldh - gel - hfj - idk$$

The solution for x is:

$$x = \frac{\Delta_x}{\Delta}$$

Repeat for y with the second column of the constant matrix replaced with the constant vector, and for z, with the third column.

While you can use Excel to solve determinants by taking products and adding/subtracting as above, Excel has function to calculate the determinate: **MDETERM**(argument), where the argument is components of a square matrix. In Excel, the to calculate the determinate of the coefficient matrix Δ:

MDTERM ((a,b,c ; d,e,f ; g,h,i))

A new row is indicated by a semi-colon.

Or, you can select a range, where range is an array of values of the matrix:
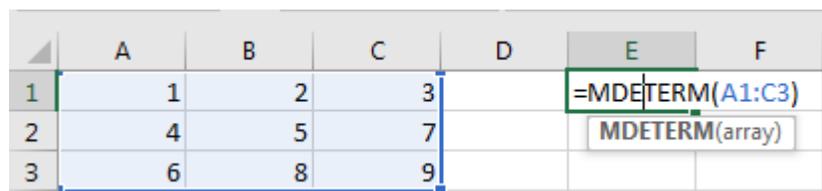
◢	A	B	C	D	E	F
1	1	2	3		=MDETERM(A1:C3)	
2	4	5	7		MDETERM(array)	
3	6	8	9			

End of Module 3